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Hyperelliptic solutions of KdV and KP equations: re-evaluation of Baker's study on hyperelliptic sigma functions

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Abstract

Explicit function forms of hyperelliptic solutions of Korteweg–de Vries (KdV) and Kadomtsev–Petviashvili (KP) equations are constructed for a given curve $y^2 = f(x)$ whose genus is three. This paper is based upon the fact that about one hundred years ago (Baker H F 1903 *Acta Math.* **27** 135–56), Baker essentially derived KdV hierarchy and KP equations by using a bilinear differential operator $\boldsymbol{D}$, identities of Pfaffians, symmetric functions, the hyperelliptic σ-function and $\wp$-functions; $\wp_{\mu\nu} = -\partial_\mu\partial_\nu \log\sigma = -(\boldsymbol{D}_\mu \boldsymbol{D}_\nu \sigma\sigma)/2\sigma^2$. The connection between his theory and the modern soliton theory is also discussed.

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1. Introduction

In this paper we will construct explicit function forms of hyperelliptic solutions of Korteweg–de Vries (KdV) and Kadomtsev–Petviashvili (KP) equations for a given curve $y^2 = f(x)$ whose genus is three, along the lines of the study of Baker's sigma function [B1, B2, B3]. This construction means re-evaluation of Baker's studies on hyperelliptic functions which were conducted one hundred years ago, in particular, his studies of algebraic functions over a general compact Riemannian surface [B3]. Although his general theory is already known to be related to Baker–Akhiezer functions [B1, K1, K2], the paper [B3] published in 1903 might have been overlooked.

According to [B3], in around 1898 he discovered series of partial differential equations which led to the hyperelliptic sigma function, σ, and $\wp$-functions, $\wp_{\mu\nu} := \partial_\mu\partial_\nu \log\sigma$. If one saw the partial differential equations, one would know that they are related to soliton equations such as the KdV equations or the KP equations. However, Baker's definition of parameters is different from that in modern soliton theory. Further as the paper [B3] requires knowledge of hyperelliptic σ- and $\wp$- functions which might not be familiar nowadays [B1, B2, O2], it is not easy to understand its contents and to confirm the derivation. In this paper, we will give correspondences between his differential equations and the KP equation and first and second equations of the KdV hierarchy in order to construct explicit function forms of their periodic multi-soliton solutions.

The identification between Baker's differential equations and these soliton equations means that Baker essentially discovered the KdV hierarchy and the KP equation one hundred years ago. In his study, he used the Pfaffian, symmetric functions, a bilinear operator $\boldsymbol{D}$, a hyperelliptic sigma function σ and $\wp$-functions; $\wp_{\mu,\nu} = -(\boldsymbol{D}_\mu \boldsymbol{D}_\nu \sigma\sigma)/2\sigma^2$.

In this paper, we will comment on its relation to soliton theory in section 4. As we will mention there, we can regard Baker's theory as being on the differentials of the first kind over a hyperelliptic curve. As compared with his theory, the ordinary soliton theories, e.g. Sato theory [SS], Date–Jimbo–Kashiwara–Miwa (DKJM) theory [DKJM], Krichever theory [K1, K2], conformal field theory [KNTY] and so on, can be considered as theories of the differentials of the second kind. Thus Baker's theory is not directly connected with the modern soliton theories, even though he used the Pfaffian, symmetric functions and a bilinear operator $\boldsymbol{D}$. Indeed he might only have been interested in properties of periodic functions on non-degenerate curves. As far as I am aware, he did not consider the soliton solutions, which are expressed by hyperbolic functions or trigonometric functions. Hence he did not arrive at Hirota's direct method [H] even though he defined and used the bilinear operator.

However, as all values appearing in Baker's theory have algorithms to evaluate themselves, we can deal with hyperelliptic functions in the framework of his theory as we can do with elliptic functions. For example, we can concretely determine any coefficients of Laurent or Taylor expansions of σ- and $\wp$-functions at any points in any hyperelliptic curves [B1, B2, B3, G, O1, O3]. Recently, requests to evaluate the hyperelliptic functions explicitly appeared from various fields, e.g. from studies on Abel functions, from number theory [G, O1, O3] and from studies of an elastica which is closely related to the KdV equations [Ma1, Ma2]. There, Baker's theory of hyperelliptic functions plays a central role [G, O1, O3, Ma2]. The purpose of this paper is to re-evaluate Baker's work from the viewpoint of soliton theory.

Only after completion of this paper did I become aware of the works of Buchstaber *et al* [BEL1, BEL2, BEL3] and others ([CEEK, EE, EEL, EEP, N] and references therein). The authors in [BEL1, BEL2, BEL3, CEEK, EE, EEL, EEP, N] also re-evaluated the theory of Baker's hyperelliptic sigma functions, which they call Kleinian functions, and have extended it from the point of view of soliton theory. For example in [B3], Baker derived a differential identity of the hyperelliptic $\wp$-functions of arbitrary genus, called fundamental formula and mentioned in section 4 of this paper, which must include the KdV hierarchy and the KP equations of higher genera but he explicitly presented them only for the genus three case. On the other hand, in [BEL1, BEL2], the authors developed a method in terms of matrices by considering a subset of $\wp$-functions $(\wp_{gi})_{\{i=1,\ldots,g\}}$ as a vector and then gave the explicit relation of the KdV hierarchy and the hyperelliptic $\wp$-functions of arbitrary genus g. Their method is consistent with the zero curvature condition in modern soliton theory. Using the hyperelliptic sigma function and defining natural sigma functions of more general algebraic curves, the authors in [BEL1, BEL2, BEL3, CEEK, EE, EEL, EEP, N] have been constructing deeper theories of Abelian functions and soliton equations. Thus, needless to say, [BEL1, BEL2, BEL3, CEEK, EE, EEL, EEP, N] are outside the realm of Baker. In fact most of results in section 2 of this paper (proposition 4 and theorem 6) has been mentioned in their studies [BEL1, BEL2, EE] and a review of part of Baker's theory in [BEL2] is very nice even for readers who are not familiar with hyperelliptic functions. In [BEL3], it was pointed out that $\wp_{11}$ of a hyperelliptic curve of genus $g > 2$ with odd degree polynomial is a solution of the KP equation, which corresponds to the relation (IV-15) in (2–15) of this paper. However in [BEL1, BEL2, BEL3, CEEK, EE, EEL, EEP, N], they did not comment upon the paper [B3], which contains interesting and fruitful results from the modern point of view as described in section 4. Further as far as I know, there has been no study on a hyperelliptic function solution of the KP equation over a hyperelliptic curve with even degree polynomial, which directly reproduces the natural dispersion relations of the KP equation. The connection between modern soliton theory [DKJM] and Baker's theory which is discussed in section 4 is also considered from the viewpoint of the re-evaluation. Thus I believe that this paper is still important.

2. Hyperelliptic solutions of KdV equations

In this section, we will consider hyperelliptic solutions of the first and second KdV equations in the KdV hierarchy. First we will define the notations and definitions that will be used in this paper. Although we mainly deal with a curve of genus three, we give definitions and expressions of hyperelliptic curves with general genus for later convenience. In this paper, we will mainly use the conventions of Ônishi [O1, O2]. We denote the set of complex numbers by $\mathbb{C}$ and the set of integers by $\mathbb{Z}$.

Notation 1. *We deal with a hyperelliptic curve X_g of genus g $(g > 0)$ given by the algebraic equation*

$$\begin{aligned} y^2 &= f(x) \\ &= \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_{2g+1} x^{2g+1} \\ &= (x - c_1)\cdots(x - c_g)(x - c_{g+1})\cdots(x - c_{2g})(x - c_{2g+1}) \end{aligned} \tag{2.1}$$

where $\lambda_{2g+1} \equiv 1$ and the λ_j and c_j are complex values.

Since we wish to treat the infinite point in this curve, we should embed it in a projective space. However as this is not difficult, we assume that the curve $y^2 = f(x)$ includes the infinite point. Further, for simplicity, we also assume that $f(x) = 0$ is not degenerate. We sometimes express a point P in the curve by the affine coordinate (x, y).

Definition 2 ([B1, p 195], [B2, p 314], [B3, p 137], [O1, pp 385–6], [O2]).

(1) Let us denote the homology of a hyperelliptic curve X_g by

$$\mathrm{H}_1(X_g, \mathbb{Z}) = \bigoplus_{j=1}^{g} \mathbb{Z}\alpha_j \oplus \bigoplus_{j=1}^{g} \mathbb{Z}\beta_j \tag{2.2}$$

where the intersections are given by $[\alpha_i, \alpha_j] = 0$, $[\beta_i, \beta_j] = 0$ and $[\alpha_i, \beta_j] = \delta_{i,j}$.

(2) The unnormalized differentials of the first kind are defined as

$$\omega_1 := \frac{\mathrm{d}x}{2y} \qquad \omega_2 := \frac{x\,\mathrm{d}x}{2y}, \ldots \qquad \omega_g := \frac{x^{g-1}\,\mathrm{d}x}{2y}. \tag{2.3}$$

(3) The unnormalized differentials of the second kind are defined as

$$\eta_j := \frac{1}{2y}\sum_{k=j}^{2g-j}(k+1-j)\lambda_{k+1+j}x^k\,\mathrm{d}x \qquad (j = 1, \ldots, g). \tag{2.4}$$

(4) The unnormalized period matrices are defined as

$$\mathbf{\Omega}' := \left[\int_{\alpha_j}\omega_i\right] \qquad \mathbf{\Omega}'' := \left[\int_{\beta_j}\omega_i\right] \qquad \mathbf{\Omega} := \begin{bmatrix}\mathbf{\Omega}' \\ \mathbf{\Omega}''\end{bmatrix}. \tag{2.5}$$

(5) The normalized period matrices are given by

$${}^t[\,\hat\omega_1 \cdots \hat\omega_g\,] := \mathbf{\Omega}'^{-1}\,{}^t[\,\omega_1 \cdots \omega_g\,] \qquad \mathbb{T} := \mathbf{\Omega}'^{-1}\mathbf{\Omega}'' \quad \hat{\mathbf{\Omega}} := \begin{bmatrix}1_g \\ \mathbb{T}\end{bmatrix}. \tag{2.6}$$

(6) The complete hyperelliptic integrals of the second kind are given by

$$H' := \left[\int_{\alpha_j} \eta_i \right] \qquad H'' := \left[\int_{\beta_j} \eta_i \right]. \tag{2.7}$$

(7) By defining the Abel map for the gth symmetric product of the curve X_g and for points $\{Q_i\}_{i=1,\ldots,g}$ in the curve:

$$\begin{aligned} \hat{w} : \mathrm{Sym}^g(X_g) \longrightarrow \mathbb{C}^g \qquad & \left(\hat{w}_k(Q_i) := \sum_{i=1}^{g} \int_{\infty}^{Q_i} \hat{\omega}_k \right) \\ w : \mathrm{Sym}^g(X_g) \longrightarrow \mathbb{C}^g \qquad & \left(w_k(Q_i) := \sum_{i=1}^{g} \int_{\infty}^{Q_i} \omega_k \right) \end{aligned} \tag{2.8}$$

the Jacobi varieties $\hat{\mathcal{J}}_g$ and $\mathcal{J}_g$ are defined as complex torus,

$$\hat{\mathcal{J}}_g := \mathbb{C}^g / \hat{\mathbf{\Lambda}} \qquad \mathcal{J}_g := \mathbb{C}^g / \mathbf{\Lambda}. \tag{2.9}$$

Here $\hat{\mathbf{\Lambda}}$ $(\mathbf{\Lambda})$ is a lattice generated by $\hat{\mathbf{\Omega}}$ $(\mathbf{\Omega})$.

(8) We define the theta function over $\mathbb{C}^g$, characterized by $\hat{\mathbf{\Lambda}}$, as

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z) := \theta\begin{bmatrix} a \\ b \end{bmatrix}(z; \mathbb{T}) := \sum_{n \in \mathbb{Z}^g} \exp\left[2\pi \mathrm{i} \left\{ \tfrac{1}{2}\, {}^t(n+a)\mathbb{T}(n+a) + {}^t(n+a)(z+b) \right\}\right] \tag{2.10}$$

for g-dimensional vectors a and b.

We should note that these contours in the integrals are, for example, given in p 3.83 in [M]. Thus the above values can be, in principle, computed in terms of a numerical method for a given $y^2 = f(x)$.

It is also noted that in (2.3), we have employed the convention of Ônishi [O1, O2], which differs from Baker's original one by a factor of 1/2. Due to the difference, the results and definitions in [B1, B2, B3] will be slightly modified but the factor set us free from extra constant factors in various situations [G, O1, O2, O3].

Definition 3 (($\wp$-function, Baker) [B1], [B2, p 336, p 358, p 370], [O1, pp 386–7], [O2]). *We prepare the coordinate in $\mathbb{C}^g$ for points $(x_i, y_i)_{i=1,\ldots,g}$ of the curve $y^2 = f(x)$,*

$$u_j := \sum_{i=1}^{g} \int_{\infty}^{(x_i, y_i)} \omega_j. \tag{2.11}$$

(1) Using the coordinate u_j, the sigma function, which is a holomorphic function over $\mathbb{C}^g$, is defined as

$$\sigma(u) = \sigma(u; X_g) := \exp(-\tfrac{1}{2}\, {}^t\, u H' \mathbf{\Omega}'^{-1} u) \vartheta \begin{bmatrix} \delta'' \\ \delta' \end{bmatrix} (\mathbf{\Omega}'^{-1} u; \mathbb{T}) \tag{2.12}$$

where

$$\delta' = {}^t\left[\, \tfrac{g}{2} \quad \tfrac{g-1}{2} \quad \cdots \quad \tfrac{1}{2} \,\right] \qquad \delta'' = {}^t\left[\, \tfrac{1}{2} \quad \cdots \quad \tfrac{1}{2} \,\right]. \tag{2.13}$$

(2) In terms of the σ-function, the $\wp$-function over the hyperelliptic curve is given by

$$\wp_{\mu\nu}(u) = -\frac{\partial^2}{\partial u_\mu \partial u_\nu} \log \sigma(u). \tag{2.14}$$

The σ-function is a well tuned theta function. Equation (2.13) is related to the so-called Riemannian constant K as mentioned on p 3.80–82 in [M]; $\delta' + \mathbb{T}\delta''$ agrees with K. As the σ-function [B2, p 336, p 358] consists of the shifting Riemann theta function (2.10) [B2, p 324, p 336], the Riemann constant K outwardly disappears. (Thus the σ-function vanishes just over the theta divisor.) Using the σ-function, Baker derived the multiple relations of $\wp$-functions and so on. Hereafter we assume that the genus of the curve is three.

Proposition 4 ([B3, pp 155–6], [O1, p 388], [O2]). *Let us express* $\wp_{\mu\nu\rho} := \partial\wp_{\mu\nu}(u)/\partial u_\rho$ *and* $\wp_{\mu\nu\rho\lambda} := \partial^2\wp_{\mu\nu}(u)/\partial u_\mu\partial u_\nu$*. Then hyperelliptic* $\wp$*-functions obey the following relations:*

$$
\begin{aligned}
&\text{(IV-1)} && \wp_{3333} - 6\wp_{33}^2 = 2\lambda_5\lambda_7 + 4\lambda_6\wp_{33} + 4\lambda_7\wp_{32}\\
&\text{(IV-2)} && \wp_{3332} - 6\wp_{33}\wp_{32} = 4\lambda_6\wp_{32} + 2\lambda_7(3\wp_{31} - \wp_{22})\\
&\text{(IV-3)} && \wp_{3331} - 6\wp_{31}\wp_{33} = 4\lambda_6\wp_{31} - 2\lambda_7\wp_{21}\\
&\text{(IV-4)} && \wp_{3322} - 4\wp_{32}^2 - 2\wp_{33}\wp_{22} = 2\lambda_5\wp_{32} + 4\lambda_6\wp_{31} - 2\lambda_7\wp_{21}\\
&\text{(IV-5)} && \wp_{3321} - 2\wp_{33}\wp_{21} - 4\wp_{32}\wp_{31} = 2\lambda_5\wp_{31}\\
&\text{(IV-6)} && \wp_{3311} - 4\wp_{31}^2 - 2\wp_{33}\wp_{11} = 2\Delta\\
&\text{(IV-7)} && \wp_{3222} - 6\wp_{32}\wp_{22} = -4\lambda_2\lambda_7 - 2\lambda_3\wp_{33} + 4\lambda_4\wp_{32} + 4\lambda_5\wp_{31} - 6\lambda_7\wp_{11}\\
&\text{(IV-8)} && \wp_{3221} - 4\wp_{32}\wp_{21} - 2\wp_{31}\wp_{22} = -2\lambda_1\lambda_7 + 4\lambda_4\wp_{31} - 2\Delta\\
&\text{(IV-9)} && \wp_{3211} - 4\wp_{31}\wp_{21} - 2\wp_{32}\wp_{11} = -4\lambda_0\lambda_7 + 2\lambda_3\wp_{31}\\
&\text{(IV-10)} && \wp_{3111} - 6\wp_{31}\wp_{11} = 4\lambda_0\wp_{33} - 2\lambda_1\wp_{32} + 4\lambda_2\wp_{31}\\
&\text{(IV-11)} && \wp_{2222} - 6\wp_{22}^2 = -8\lambda_2\lambda_6 + 2\lambda_3\lambda_5\\
&&& \qquad -6\lambda_1\lambda_7 - 12\lambda_2\wp_{33} + 4\lambda_3\wp_{32} + 4\lambda_4\wp_{22} + 4\lambda_5\wp_{21} - 12\lambda_6\wp_{11} + 12\Delta\\
&\text{(IV-12)} && \wp_{2221} - 6\wp_{22}\wp_{21} = -4\lambda_1\lambda_6 - 8\lambda_0\lambda_7 - 6\lambda_1\wp_{33} + 4\lambda_3\wp_{31} + 4\lambda_4\wp_{21} - 2\lambda_5\wp_{11}\\
&\text{(IV-13)} && \wp_{2211} - 4\wp_{21}^2 - 2\wp_{22}\wp_{11} = -8\lambda_0\lambda_6 - 8\lambda_0\wp_{33} - 2\lambda_1\wp_{32} + 4\lambda_2\wp_{31} + 2\lambda_3\wp_{21}\\
&\text{(IV-14)} && \wp_{2111} - 6\wp_{21}\wp_{11} = -2\lambda_0\lambda_5 - 8\lambda_0\wp_{32} + 2\lambda_1(3\wp_{31} - \wp_{22}) + 4\lambda_2\wp_{21}\\
&\text{(IV-15)} && \wp_{1111} - 6\wp_{11}^2 = -4\lambda_0\lambda_4 + 2\lambda_1\lambda_3 + 4\lambda_0(4\wp_{31} - 3\wp_{22}) + 4\lambda_1\wp_{21} + 4\lambda_2\wp_{11}
\end{aligned}
\tag{2.15}
$$

where

$$\Delta = \wp_{32}\wp_{21} - \wp_{31}\wp_{22} + \wp_{31}^2 - \wp_{33}\wp_{11}. \tag{2.16}$$

Remark 5.

(1) Due to the definitions, indices of $\wp$ are symmetric, i.e. $\wp_{\mu\nu} = \wp_{\nu\mu}$, $\wp_{\mu\nu\rho} = \wp_{\rho\mu\nu} = \wp_{\nu\rho\mu}$ and so on.

(2) The above equations are independent because the axes of the Jacobian $\mathcal{J}_g$ are independent.

(3) In the same manner as Baker [B3, p 151], by introducing the bilinear differential operator $\boldsymbol{D}_\nu$,

$$\boldsymbol{D}_\mu\sigma(u)\sigma(u) := \left(\frac{\partial}{\partial u'_\mu} - \frac{\partial}{\partial u_\mu}\right)\sigma(u')\sigma(u)|_{u=u'} \tag{2.17}$$

we have the relations

$$\wp_{\mu\nu} = -\frac{1}{2\sigma^2}\boldsymbol{D}_\mu\boldsymbol{D}_\nu\sigma\sigma \tag{2.18}$$

$$\wp_{\lambda\mu\nu\rho} - 2(\wp_{\mu\nu}\wp_{\lambda\rho} + \wp_{\nu\lambda}\wp_{\rho\mu} + \wp_{\lambda\mu}\wp_{\rho\nu}) = -\frac{1}{2\sigma^2}\boldsymbol{D}_\lambda\boldsymbol{D}_\mu\boldsymbol{D}_\nu\boldsymbol{D}_\rho\sigma\sigma. \tag{2.19}$$

Then the equations in proposition 4 can be regarded as the bilinear equations of σ-functions. For example, (IV-1) is given by

$$(\boldsymbol{D}_3^4 - 4\lambda_6\boldsymbol{D}_3^2 - 4\boldsymbol{D}_3\boldsymbol{D}_2 - 4\lambda_5\lambda_7)\sigma\sigma = 0. \tag{2.20}$$

Theorem 6. *For $v = -2(\wp_{33} + \lambda_6/3)$ and $v(t_1, t_3, t_5) = v(u_3, -\frac{u_2}{2^2}, \frac{u_1}{2^4} + \frac{3}{2^4\lambda_6}u_2)$ both the v obey first and second KdV equations:*

$$\partial_{t_3} v + 6v\partial_{t_1} v + \partial_{t_1}^3 v = 0 \tag{2.21}$$

$$\partial_{t_5} v + 30v^2\partial_{t_1} v + 20\partial_{t_1} v\partial_{t_1}^2 v + 10v\partial_{t_1}^3 v + \partial_{t_1}^5 v = 0. \tag{2.22}$$

Proof. By differentiating (IV-1) in u_3 and tuning them, we obtain the KdV equation. We note that the second KdV equation is expressed by

$$\partial_{t_5} v + (\partial_{t_1}^2 + 2v + 2\partial_{t_1} v\partial_{t_1}^{-1})(6v\partial_{t_1} v + \partial_{t_1}^3 v) = 0 \tag{2.23}$$

where $\partial_{t_1}^{-1}$ implies an integral with respect to t_1. By setting $2\partial_{u_3} \times$ (IV-2) + $\partial_{u_2} \times$ (IV-1) and $\partial_{t_5} = 16\partial_{u_1} + \frac{16\lambda_2}{3}\partial_{u_2}$, we obtain second KdV equation. □

Remark 7.

(1) Theorem 6 and the definition of $\wp$ mean that solutions of the KdV equation are explicitly constructed. The quantities in definitions 2 and 3 can be, in principle, evaluated in terms of numerical computations because there is no ambiguous parameter.

(2) We note the dispersion relations: u_j behaves like $(1/\overline{x})^{2(g-j)+1}$ around the infinity point if we use the local coordinate $\overline{x}^2 := x$. By comparing the order of $\overline{x}$, denoted by $\mathrm{ord}_{\overline{x}}$, we have the relations

$$\mathrm{ord}_{\overline{x}}(u_2) = 3\mathrm{ord}_{\overline{x}}(u_3) \qquad \mathrm{ord}_{\overline{x}}(u_1) = 5\mathrm{ord}_{\overline{x}}(u_3). \tag{2.24}$$

These are the dispersion relations of the KdV equations.

(3) Roughly speaking, integrating the KdV equation in t_1 becomes (IV-1) in proposition 4. Then there appears an undetermined integral constant. However in proposition 4, it is fixed and associated with the coefficients of the algebraic equation $y^2 = f(x)$. Thus (IV-1) in proposition 4 is more fundamental than the KdV equation.

(4) For the genus two case: we put $\partial\sigma/\partial u_3 = 0$ and $\lambda_6 = \lambda_7 = 0$; (IV-1)–(IV-10) becomes meaningless as $0 = 0$ and $\Delta = 0$. $v = -2(\wp_{22} + \lambda_4/3)$ and $v(t_1, t_3) = v(u_2, -\frac{u_1}{2^2})$ obey the first KdV equation (2.21).

(5) For the genus one case or elliptic functions case: we put $\partial\sigma/\partial u_\mu = 0$ ($\mu = 2, 3$) and $\lambda_a = 0$ ($a = 4, 5, 6, 7$); only (IV-15) survives, which is the relation of the elliptic $\wp$-function.

3. Hyperelliptic solutions of the KP equation

Instead of the curve of $(2g+1)$-degree, we will deal with a hyperelliptic curve of $(2g+2)$-degree in this section.

Notation 8.

$$\begin{aligned} y^2 = \overline{f}(x) &= \bar{\lambda}_0 + \bar{\lambda}_1 x + \bar{\lambda}_2 x^2 + \cdots + \bar{\lambda}_{2g+2} x^{2g+2} \\ &= (x - \alpha_1)\cdots(x - \alpha_g)(x - \alpha_{g+1})\cdots(x - \alpha_{2g})(x - \alpha_{2g+1})(x - \alpha_{2g+2}) \end{aligned} \tag{3.1}$$

where $\bar{\lambda}_{2g+2} \equiv 1$ and the $\bar{\lambda}_j$ and α_j are complex values.

Remark 9 ([B1, p 195], [B3, pp 144–5]).

(1) The transformation between $y^2 = f(x)$ and $\zeta^2 = \overline{f}(\xi)$ is as follows:

$$x = \frac{a}{\xi - \alpha_{2g+2}} \qquad c_i = \frac{a}{\alpha_i - \alpha_{2g+2}} \qquad \zeta = \frac{(\xi - \alpha_{2g+2})^{g+1}}{-4\prod_i^{2g-1} c_j} y. \tag{3.2}$$

(2) The unnormalized differentials of the first kind are defined by

$$\omega_1 = \frac{dx}{2y} \qquad \omega_2 = \frac{x\,dx}{2y} \qquad \cdots \qquad \omega_g = \frac{x^{g-1}\,dx}{2y}. \tag{3.3}$$

(3) The unnormalized differentials of the second kind are defined by [B2, p 195]

$$\eta_j = \frac{1}{2y}\sum_{k=j}^{2g+1-j}(k+1-j)\bar{\lambda}_{k+1+j}x^k\,dx \qquad (j=1,\ldots,g). \tag{3.4}$$

(4) The definition of σ- and $\wp$-functions are the same as those in definitions 2 and 3, where we regard that y obeys the equation $y^2 = \overline{f}(x)$ instead of $y^2 = f(x)$.

Proposition 10 ([B3, pp 155–6]). *The hyperelliptic $\wp$-functions of a curve $y^2 = \overline{f}(x)$ $(g = 3)$ obey the following relations:*

$$\begin{array}{ll}
\text{(X-1)} & \wp_{3333} - 6\wp_{33}^2 = 2\overline{\lambda}_5\overline{\lambda}_7 + 4\overline{\lambda}_6\wp_{33} + 4\overline{\lambda}_7\wp_{32} - 8\overline{\lambda}_4\overline{\lambda}_8 + 4\overline{\lambda}_8(4\wp_{31} - 3\wp_{22}) \\
\text{(X-2)} & \wp_{3332} - 6\wp_{33}\wp_{32} = 4\overline{\lambda}_6\wp_{32} + 2\overline{\lambda}_7(3\wp_{31} - \wp_{22}) - 4\overline{\lambda}_3\overline{\lambda}_8 + 8\overline{\lambda}_8\wp_{21} \\
\text{(X-3)} & \wp_{3331} - 6\wp_{31}\wp_{33} = 4\overline{\lambda}_6\wp_{31} - 2\overline{\lambda}_7\wp_{21} + 4\overline{\lambda}_8\wp_{11} \\
\text{(X-4)} & \wp_{3322} - 4\wp_{32}^2 - 2\wp_{33}\wp_{22} = 2\overline{\lambda}_5\wp_{32} + 4\overline{\lambda}_6\wp_{31} - 2\overline{\lambda}_7\wp_{21} - 8\overline{\lambda}_2\overline{\lambda}_8 - 8\overline{\lambda}_8\wp_{11} \\
\text{(X-5)} & \wp_{3321} - 2\wp_{33}\wp_{21} - 4\wp_{32}\wp_{31} = 2\overline{\lambda}_5\wp_{31} - 4\overline{\lambda}_1\overline{\lambda}_8 \\
\text{(X-6)} & \wp_{3311} - 4\wp_{31}^2 - 2\wp_{33}\wp_{11} = 2\Delta \\
\text{(X-7)} & \wp_{3222} - 6\wp_{32}\wp_{22} = -4\overline{\lambda}_2\overline{\lambda}_7 - 2\overline{\lambda}_3\wp_{33} + 4\overline{\lambda}_4\wp_{32} + 4\overline{\lambda}_5\wp_{31} - 6\overline{\lambda}_7\wp_{11} - 8\overline{\lambda}_1\overline{\lambda}_8 \\
\text{(X-8)} & \wp_{3221} - 4\wp_{32}\wp_{21} - 2\wp_{31}\wp_{22} = -2\overline{\lambda}_1\overline{\lambda}_7 + 4\overline{\lambda}_4\wp_{31} - 2\Delta - 8\overline{\lambda}_0\overline{\lambda}_8
\end{array} \tag{3.5}$$

together with relations (X-9)–(X-15) and Δ which have the same form as those in proposition 4 by replacing the λ with the $\overline{\lambda}$.

Theorem 11. *For $v = -2(\wp_{33} + \overline{\lambda}_6/3)$ and $u(t_1, t_2, t_3) = v(u_3, \frac{u_2}{2\sqrt{-3}}, -\frac{u_1}{2^4} - \frac{3}{2^2\overline{\lambda}_7}u_2)$ both the v obey the KP equation*

$$\partial_{t_1}(\partial_{t_3}v + 6v\partial_{t_1}v + \partial_{t_1}^3 v) = \partial_{t_2}^2 v. \tag{3.6}$$

Proof. Noting that $\overline{\lambda}_8 = 1$, direct substitution of the v into (3.6) gives the differential of (X-1) in u_3. □

Remark 12.

(1) Theorem 11 means that we obtain an explicit function form of the hyperelliptic function solution of the KP equation.

(2) We note the dispersion relation. Since the curve $y^2 = \overline{f}(x)$ is not ramified at the infinity point, u_j behaves like $(1/x)^{(g-j)}$ up to a constant factor. By comparing the order of x, denoted by ord_x, we have the relations

$$\mathrm{ord}_x(u_2) = 2\mathrm{ord}_x(u_3) \qquad \mathrm{ord}_x(u_1) = 3\mathrm{ord}_x(u_3). \tag{3.7}$$

These are the dispersion relations of the KP equation.

4. Discussion

Since the derivation of proposition 10 is essentially the same as that of proposition 4, we will only give a sketch of the derivation of the differential equations in proposition 4 and comment upon its relation to the soliton theory.

Definition 13 ([B1, p 195], [B2, p 314, pp 335-06], [O2]). *For points of* $\mathrm{P}(x, y)$*,* $\mathrm{Q}(z, w)$*,* $\mathrm{A}(a, b)$*,* $\mathrm{B}(c, d)$ *over* X_g*, we introduce the following quantities:*

(1)

$$\boldsymbol{R}_{\mathrm{Q,B}}^{\mathrm{P,A}} := \int_{\mathrm{A}}^{\mathrm{P}} \int_{\mathrm{B}}^{\mathrm{Q}} \frac{f(x,z) + 2yw}{(x-z)^2} \frac{\mathrm{d}x}{2y} \frac{\mathrm{d}z}{2w} \tag{4.1}$$

where

$$f(x,z) := \sum_{j=0}^{g} x^j z^j (\lambda_{2j+1}(x+z) + 2\lambda_{2j}). \tag{4.2}$$

(2)

$$\boldsymbol{P}_{\mathrm{Q,B}}^{\mathrm{P,A}} := \int_{\mathrm{A}}^{\mathrm{P}} \left(\frac{y+w}{x-z} - \frac{y+\mathrm{d}}{x-c} \right) \frac{\mathrm{d}x}{2y}. \tag{4.3}$$

Proposition 14 ([B1, pp 194–5], [B2, p 318, p 336], [O2]).

(1) $\boldsymbol{R}_{\mathrm{Q,B}}^{\mathrm{P,A}}$ *and* $\boldsymbol{P}_{\mathrm{Q,B}}^{\mathrm{P,A}}$ *as functions of* P *have singularity around* $\mathrm{P} = \mathrm{Q}, \mathrm{B}$ *of first order with the residues* $1, -1$ *and are holomorphic otherwise. In other words, they are unnormalized third differentials.*

(2)

$$\boldsymbol{R}_{\mathrm{Q,B}}^{\mathrm{P,A}} = \int_{\mathrm{A}}^{\mathrm{P}} \omega_1 \int_{\mathrm{B}}^{\mathrm{Q}} \eta_1 + \cdots + \int_{\mathrm{A}}^{\mathrm{P}} \omega_g \int_{\mathrm{B}}^{\mathrm{Q}} \eta_g + \boldsymbol{P}_{\mathrm{Q,B}}^{\mathrm{P,A}}. \tag{4.4}$$

(3) For $\mathrm{P}_j, \mathrm{Q}_j \in X$*,* $(j = 1, \ldots, g)$*, and*

$$u = \sum_{j=1}^{g} \int_{\infty}^{\mathrm{P}_j} \omega \qquad u' = \sum_{j=1}^{g} \int_{\infty}^{\mathrm{Q}_j} \omega \tag{4.5}$$

the following relation holds:

$$\exp\left(\sum_{j=1}^{g} \boldsymbol{R}_{\overline{\mathrm{P}_j},\overline{\mathrm{Q}_j}}^{\mathrm{P,Q}} \right) = \frac{\sigma\left(\int_{\infty}^{\mathrm{P}} \omega + u\right) \sigma\left(\int_{\infty}^{\mathrm{Q}} \omega + u'\right)}{\sigma\left(\int_{\infty}^{\mathrm{P}} \omega + u'\right) \sigma\left(\int_{\infty}^{\mathrm{Q}} \omega + u\right)} \tag{4.6}$$

where $\overline{\mathrm{P}_j}$ $(\overline{\mathrm{Q}_j})$ *is the conjugate of* P_j (Q_j) *with respect to the symmetry of the hyperelliptic curve* $(x, y) \to (x, -y)$.

Remark 15. The relation (4.6) is very important. It holds for appropriate σ-functions and third differentials in a general compact Riemannian surface [B1, p 290], even though their form cannot be globally written like definition 13. As we will show below, the relation plays important roles in both Baker's theory and DKJM-theory [DKJM].

Here we will sketch the derivation of the equations in proposition 4 following [B1] and [B3]. First we introduce the variables for the divisors $\mathrm{P}_j = (x_j, y_j)$ and $\mathrm{P} = (x, y) \equiv (x_0, y_0)$ in the notation of proposition 14 (3),

$$\mathrm{t} := \left(\int_\infty^P \omega + u \right) \tag{4.7}$$

$$R(z) := (z - x_0)F(z) := (z - x_0)(z - x_1)(z - x_2)\cdots(z - x_g) \tag{4.8}$$

$$\frac{R(z)}{(z - x_r)(z - x_s)} =: z^{g-1} + c_1^{r,s} z^{g-2} + c_2^{r,s} z^{g-3} + \cdots + c_g^{r,s} \tag{4.9}$$

and for the generic parameter e,

$$\overline{\delta}_e := \sum_{\mu=1}^{g} e^{\mu-1} \frac{\partial}{\partial \mathrm{t}_\mu}. \tag{4.10}$$

We operate $\overline{\delta}_{e_1}\overline{\delta}_{e_2}$ on both sides of relation (4.6) in proposition 14. We should note the relation

$$\sum_{r=0, r\neq s}^{g} \frac{x_r - x_s}{R'(x_r)} c_{l-1}^{r,s} x_r^{g-k} = \delta_l^k \tag{4.11}$$

where $c_0^{r,s} = 1$ and $R'(x_r) = \mathrm{d}R(z)/\mathrm{d}z|_{z=x_r}$. By taking the limit $x_0 \to \infty$, we obtain [B1, p 328, p 376]

$$\sum_{\lambda=1}^{g}\sum_{\mu=1}^{g} \wp_{\lambda\mu}(u) e_1^{\lambda-1} e_2^{\mu-1} = \left(\sum_{r=1,s=1}^{g} \frac{F(e_1)F(e_2)(2y_r y_s - f(x_r, x_s))}{(e_1 - x_r)(e_2 - x_r)(e_1 - x_s)(e_2 - x_s)F'(x_r)F'(x_s)} \right). \tag{4.12}$$

We deform it to obtain [B1, p 328], [B3, p 138],

$$\begin{aligned} \sum_{\lambda=1}^{g}\sum_{\mu=1}^{g} \wp_{\lambda\mu}(u) e_1^{\lambda-1} e_2^{\mu-1} &= F(e_1)F(e_2) \left(\sum_{r=1}^{g} \frac{y_r}{(e_1 - x_r)(e_2 - x_r)F'(x_r)} \right)^2 \\ &\quad - \frac{f(e_1)F(e_2)}{(e_1 - e_2)^2 F(e_1)} - \frac{f(e_2)F(e_1)}{(e_1 - e_2)^2 F(e_2)} + \frac{f(e_1, e_2)}{(e_1 - e_2)^2}. \end{aligned} \tag{4.13}$$

Even though in [B3] Baker adopted this formula (4.13) as a definition of $\wp$-functions, his arguments on this formula were derived from a number of studies on the hyperelliptic function [B1,B2]. Thus we should regard (4.13) as a theorem which was proved in [B1].

Introducing another operator,

$$\delta_e = \frac{1}{F(e)} \sum_{j=1}^{g} \mathrm{e}^{j-1} \frac{\partial}{\partial u_j} \tag{4.14}$$

we operate $\delta_{e_3}\delta_{e_4}$ on the above relation (4.13) and then we have the 'fundamental formula' [B3, p 144]. Section I in [B3] is devoted to the derivation of his fundamental formula, which is very tedious and complex but somewhat attractive. In fact, tracing his derivations makes me feel that there might be deep symmetry behind his theory. In section II in [B3], Baker concentrated on the genus three case. By comparing the coefficients of each $e_1^a e_2^b e_3^c e_4^d$, he discovered the differential equations in propositions 4 and 10. In the comparison, Baker used the symmetric functions, Pfaffian and bilinear operators. The symmetric functions naturally appear because the differential of the first kind in the hyperelliptic curve is expressed by [B3]

$$\begin{pmatrix} \mathrm{d}u_1 \\ \mathrm{d}u_2 \\ \mathrm{d}u_3 \\ \cdot \\ \cdot \\ \mathrm{d}u_g \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1/y_1 & 1/y_2 & \cdots & 1/y_g \\ x_1/y_1 & x_2/y_2 & \cdots & x_g/y_g \\ x_1^2/y_1 & x_2^2/y_2 & \cdots & x_g^2/y_g \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ x_1^{g-1}/y_1 & x_2^{g-1}/y_2 & \cdots & x_g^{g-1}/y_g \end{pmatrix} \begin{pmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \\ \mathrm{d}x_3 \\ \cdot \\ \cdot \\ \mathrm{d}x_g \end{pmatrix}. \tag{4.15}$$

This matrix resembles the Vandermonde matrix. In fact (4.11) is an identity used in the construction of the inverse matrix of the Vandermonde matrix.

Corresponding to the above matrix (4.15), the behaviour of differentials of the second kind in the theory of KP hierarchy [K1, K2, SS, DKJM, KNTY] is sometimes determined by the Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \overline{x}_1 & \overline{x}_2 & \cdots & \overline{x}_p \\ \overline{x}_1^2 & \overline{x}_2^2 & \cdots & \overline{x}_p^2 \\ \cdot & \cdot & \cdots & \cdot \\ \overline{x}_1^{p-1} & \overline{x}_2^{p-1} & \cdots & \overline{x}_p^{p-1} \end{pmatrix}. \tag{4.16}$$

The difference between Baker's theory of hyperelliptic function and modern soliton theory could be regarded as the difference between (4.15) and (4.16).

In modern soliton theory [SS, DKJM, KNTY], we deal with a formal graded ring $G\mathbb{C}[[\overline{x}]] := \cup_n G^n\mathbb{C}[[\overline{x}]]$ related to the degrees of the $\overline{x}$ as a localized ring at the infinity point of an algebraic curve. Then we consider maps among quotient modules $G^n\mathbb{C}[[\overline{x}]]/G^{n-1}\mathbb{C}[[\overline{x}]]$, which consist of $\partial_{\overline{x}}$ and $\overline{x}$. The differential ring generated by $\partial_{\overline{x}}$ and $\overline{x}$ becomes Sato's theory [SS] and conformal field theory [KNTY] after appropriately modifying it. There naturally appear the Vandermonde matrix (4.16) of the $\overline{x}$, symmetric functions, Pfaffian related to the behaviour of the differential of the second kind around the infinity point; the Vandermonde determinate is related to fermion amplitude [DKJM, KNTY].

In the theory of differentials of the second kind, when one determines the global behaviour of an algebraic function on a curve by its local data around the infinity point, Baker uses the properties of holomorphic functions over the curves, such as existence theorem, flabby of related sheaves and so on. On the other hand, Baker's theory is of differentials of the first kind and it is a global theory because differentials of the first kind are holomorphic all over the curve and are given explicitly. Accordingly, we can deal with the hyperelliptic functions in the framework of Baker's theory as we do with elliptic functions.

We will comment on proposition 4 in the framework of DKJM-theory [DKJM].

Remark 16. For points $\mathrm{P} = (x, y)$, $\mathrm{Q} = (\sqrt{-1}x, y)$ and $\overline{\mathrm{P}} = (x, -y)$ around the infinity points $x = \overline{x}^2$, we obtain the following relations:

(1)
$$\boldsymbol{R}^{\mathrm{P,Q}}_{\overline{\mathrm{A}},\overline{\mathrm{B}}} = \boldsymbol{R}^{\overline{\mathrm{A}},\overline{\mathrm{B}}}_{\mathrm{P,Q}}. \tag{4.17}$$

(2)
$$\int_\infty^{(x,y)} \omega_\mu = -\int_\infty^{(\sqrt{-1}x,y)} \omega_\mu = \int_{(\sqrt{-1}x,y)}^{(x,y)} \frac{x^{\mu-1}\,\mathrm{d}x}{y}$$
$$= -\frac{1}{2g-2\mu+1}\frac{1}{\overline{x}^{2g-2\mu+1}} + \text{lower order terms}. \tag{4.18}$$

(3)
$$\int_{(\sqrt{-1}x,y)}^{(x,y)} \eta_j = 2[\overline{x}^{2g-2j+1}] + \text{lower order terms}. \tag{4.19}$$

(4) $$\sum_{j=1}^{g} \boldsymbol{R}^{\mathrm{P,Q}}_{\overline{\mathrm{P}_j,\mathrm{Q}_j}} = -2[(u_1 - u_1')\overline{x}^{2g-1} + (u_2 - u_2')\overline{x}^{2g-3} + \cdots + (u_g - u_g')\overline{x}] + \sum_{j=1}^{g} \boldsymbol{P}^{\mathrm{P,Q}}_{\overline{\mathrm{P}_j,\mathrm{Q}_j}} + \text{lower order terms}. \tag{4.20}$$

Using remark 16, setting $g = \infty$ and neglecting the lower-order terms, relation (3) in proposition 14 is reduced to the generating relation of the KdV hierarchy in the DKJM method:

$$\oint_{\infty} \frac{\mathrm{d}\overline{x}}{\overline{x}} \exp\left(\sum_{j=1}^{g}(u_j - u_j')\overline{x}^{2g-2\mathrm{i}+1}\right) \times\sigma\left(u_1 - \frac{1}{2g-1}\frac{1}{\overline{x}^{2g-1}}, u_2 - \frac{1}{2g-3}\frac{1}{\overline{x}^{2g-3}}, \ldots, u_g - \frac{1}{\overline{x}}\right) \times\sigma\left(u_1' + \frac{1}{2g-1}\frac{1}{\overline{x}^{2g-1}}, u_2' + \frac{1}{2g-3}\frac{1}{\overline{x}^{2g-3}}, \ldots, u_g' + \frac{1}{\overline{x}}\right) = 0. \tag{4.21}$$

In terms of differential operators, we can rewrite this relation and then we obtain the KdV hierarchy [DKJM]. Thus the origins of the KdV hierarchy in Baker's method and the DKJM method are the same.

Remark 17. We will now summarize the differences between soliton theory and Baker's theory.

(1) As in soliton theory of the KdV hierarchy [DKJM, K1, K2, SS], we investigate the behaviour of meromorophic functions around the infinity point of a hyperelliptic curve: (i) it can be regarded as a theory of differentials of the second kind; (ii) it can be extended to the theory of meromorophic functions of a general compact Riemannian surface as the theory of the KP hierarchy [DKJM, K1, K2, KNTY, SS]; and (iii) we cannot determine fine structure of meromorophic functions of non-degenerate curves.

(2) As in the Baker's theory of hyperelliptic $\wp$-functions, we consider the behaviour of $\wp$-functions around generic points $(x_1, y_1), \ldots, (x_g, y_g)$ of a hyperelliptic curve: (i) we directly deal with differentials of the first kind which are holomorphic all over the curve; (ii) we can determine all parameters in $\wp$-functions of the curve; (iii) we can give explicit function forms of $\wp$-functions and coefficients of Laurent expansions around any points in the curve; and (iv) we cannot extend it to a general compact Riemannian surface with this concreteness.

(3) The differentials of the first and second kinds are complementarily connected as the term in (4.4) of the most important identity (4.6). Thus in (4.6), they behave like two sides of the same coin.

Finally we comment upon this study. In Baker's theory, we have no ambiguous and dependent parameters while in ordinary soliton theory of periodic solutions there appear undetermined parameters which must satisfy several relations. Hirota and Ito gave explicit function forms of hyperelliptic functions of genera two and three as periodic solutions of the KdV equation (2.21) [HI]; they determined several parameters by means of numerical computations. However functions should be expressed only by independent variables and thus Baker's theory has the advantage and is appropriate even from the viewpoint of numerical study. I hope that in the near future, anyone would be able to plot graphs of any hyperelliptic functions or any periodic multi-soliton solutions like the graphs in [HI], using a personal computer and a laser printer, as we can do for elliptic functions or elliptic soliton solutions.

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